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THE ENERGY OF SINGULAR EFFECTS IN AN ELASTIC ANISOTROPIC MEDIUM†

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The problem of determining the accumulated energy function, the strain energy and the energy concentration coefficient corresponding to singularities in the form of concentrated forces and moments in an infinite homogeneous linearly elastic anisotropic medium is considered. The method of multipole expansions is used. Examples are presented. © 1997 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

For a hyperelastic homogeneous anisotropic medium, the accumulated energy function can be determined as a bilinear form in a space of second rank symmetric tensors which correspond to linear deformations of the medium [1]

$$f(\mathbf{x}) \equiv \tfrac{1}{2}\varepsilon(\mathbf{x}) \cdot\cdot \mathbf{C} \cdot\cdot \varepsilon(\mathbf{x}), \quad \mathbf{x} \in R^3 \tag{1.1}$$

where ε is the strain tensor field and $\mathbf{C}$ is a tetravalent elasticity tensor which characterizes the elastic properties of the medium. It is assumed that the tensor $\mathbf{C}$ is strictly elliptic such that $f > 0$ when $\varepsilon(\mathbf{x}) \neq 0$. The accumulated energy function is of interest when analysing the redistribution of strain energy in different directions during the loading of an anisotropic medium.

A double zero-moment force, a double force with a moment and a dilation centre [1] located at the origin of coordinates of an infinite medium are considered as the external effects. It is essential that, in the case of these types of effects, the accumulated energy function turns out to be positively homogeneous with respect to the coordinates and only has a singularity at the origin of the coordinate system.

When the accumulated energy function is known, it is also possible to calculate the strain energy

$$W_\delta = \int_{R^3 \setminus B_\delta} f(\mathbf{x})dx \tag{1.2}$$

where B_δ is a sphere of radius δ with centre at the origin of the coordinate system, and dx is the Lebesgue measure in R^3. The need to exclude the neighbourhood of the coordinate origin in (1.2), where the concentrated force effects are applied, is due, as is shown below, to the existence of a non-integrable singularity of the function f. The separating out of neighbourhoods containing non-integrable singularities when determining the strain energy has also been encountered previously in problems in dislocation theory, for example.

Together with the energy which is calculated using formula (1.2), we are also interested in the magnitude of the energy concentration coefficient, defined by the formula

$$W_0 = \lim_{\delta \to 0} \delta^\alpha W_\delta \tag{1.3}$$

where α is the asymptotic exponent in formula (1.2) when $\delta \to 0$.

The multipole expansion method [2] is used to obtain an analytic expression for the function f, which corresponds to the types of effects under consideration. This enables us to represent f in the form of an absolutely convergent series in spherical harmonics. The same method is used to calculate the strain energy.

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2. BASIC RELATIONS

The Lamé equilibrium equations for an anisotropic medium can be represented in the form

$$\mathbf{A}(\partial_x)\mathbf{u}(\mathbf{x}) \equiv -\mathrm{div}_x \mathbf{C}\cdot\cdot\nabla_x \mathbf{u}(\mathbf{x}) = 0 \tag{2.1}$$

where $\mathbf{A}$ is the matrix differential operator of the equilibrium equations and $\mathbf{u}$ is the vector field of the displacements in the medium. Application of the Fourier integral transform

$$g^\wedge(\xi) = \int_{R^3} g(\mathbf{x})\exp(-2\pi i\mathbf{x}\cdot\xi)dx$$

to (2.1) enables us to obtain the symbol of the equilibrium equation operator and, using this, the symbol of the fundamental solution of the equilibrium equations can be written in the form

$$\mathbf{E}^\wedge(\xi) = \mathbf{A}_0^\wedge(\xi)/\det\mathbf{A}^\wedge(\xi), \quad \mathbf{A}^\wedge(\xi) = (2\pi)^2\xi\cdot\mathbf{C}\cdot\xi \tag{2.2}$$

where $\mathbf{A}_0^\wedge$ is the matrix of the cofactors of the symbol $\mathbf{A}^\wedge$. Formula (2.2) shows that the symbol $\mathbf{E}^\wedge$ is positively homogeneous with respect to $|\xi|$ and of degree –2.

In the case of general anisotropy, the fundamental solution of Eqs (2.1), associated with the inverse Fourier transformation of expression (2.2), can only be obtained numerically.

3. SINGULAR EFFECTS

The following types of individual concentrated force effects in an infinite medium and the (Fourier-transformed) displacement and deformation fields caused by them are considered.

(a) *A double zero-moment force*

$$\mathbf{u}^\wedge(\xi) = 2\pi i(\mathbf{n}\cdot\xi)\mathbf{E}^\wedge(\xi)\cdot\mathbf{n}, \quad \epsilon^\wedge(\xi) = -(2\pi)^2(\mathbf{n}\cdot\xi)\xi\otimes\mathbf{E}^\wedge(\xi)\cdot\mathbf{n} \tag{3.1}$$

where $\mathbf{n}$ is the unit vector directed along the line of action of the double force.

(b) *A double force with a moment.* The corresponding expressions differ from (3.1) in the replacement of $(\mathbf{n}\cdot\xi)$ by $(\mathbf{n}_\perp\cdot\xi)$ where $\mathbf{n}_\perp$ is the direction of the "arm" of the pair and $\mathbf{n}$ is the direction of one of the forces.

(c) *An extension–compression centre*

$$\mathbf{u}^\wedge(\xi) = 2\pi i\mathbf{E}^\wedge(\xi)\cdot\xi, \quad \epsilon^\wedge(\xi) = -(2\pi)^2\,\xi\otimes\mathbf{E}^\wedge(\xi)\cdot\xi \tag{3.2}$$

4. THE MULTIPOLE EXPANSION METHOD

An analysis of expressions (3.1) and (3.2) for the strain tensor shows that, in each of the cases considered, the deformations $\epsilon^\wedge$ turn out to be positively homogeneous of degree 0 with respect to $|\xi|$ and real-analytic with respect to ξ everywhere in $R^3\backslash 0$. By analogy with [2], we shall expand them in series in surface spherical harmonics

$$\epsilon^\wedge(\xi) = \sum_{n=0,2,4,\ldots}\sum_{p=1}^{2n+1}\epsilon^{np}Y_n^p(\xi'), \quad \xi' = \frac{\xi}{|\xi|} \tag{4.1}$$

where ϵ^{np} are the coefficients of the multipole series, determined by integration of the tensor $\epsilon^\wedge$ over a unit sphere S in R^3 and Y_n^p are spherical harmonics of degree n and index p. The summation in (4.1) is carried out over the spherical harmonics of even order n in view of the evenness of $\epsilon^\wedge$ on a unit sphere. It is essential that the series on the right-hand side of (4.1) converge absolutely on S.

The inverse Fourier transformation which is applied to the right-hand side of (4.1) enables one to obtain the required strain tensors in the space of the originals also in the form of multipolar series

$$\epsilon(\mathbf{x}) = |\mathbf{x}|^{-3}\sum_{n=2,4,\ldots}\gamma_n\sum_{p=1}^{2n+1}\epsilon^{np}Y_n^p(\mathbf{x}'), \quad \mathbf{x}' = \frac{\mathbf{x}}{|\mathbf{x}|}, \quad \gamma_n = i^n\pi^{-3/2}\Gamma\left(\frac{n+3}{2}\right)/\Gamma\left(\frac{n}{2}\right)$$

where γ_n are multipliers for the transition from the space of the images to the initial space [2]. The

zeroth order harmonic is excluded from the summation in (4.2) as it would lead to a δ-like component in the expression for the strain tensor.

5. THE STRAIN ENERGY

Substitution of expansion (4.2) into the expression for the accumulated energy function (1.1) gives

$$f(\mathbf{x}) = \frac{1}{2}|\mathbf{x}|^{-6} \sum_{n=2,4,\ldots} \sum_{m=2,4,\ldots} \gamma_n \gamma_m \sum_{p=1}^{2n+1} \sum_{q=1}^{2m+1} \boldsymbol{\epsilon}^{np} \cdot\cdot \mathbf{C} \cdot\cdot \boldsymbol{\epsilon}^{mq} Y_n^p(\mathbf{x}') Y_m^q(\mathbf{x}') \tag{5.1}$$

Formula (5.1) shows that the function f has a singularity at the origin of coordinates of the form $O(|\mathbf{x}|^{-6})$ such that energy integral turns out to be divergent.

The separating out of the spherical neighbourhood of the origin of coordinates in (1.2) enables one to obtain finite values for the energy integral. When account is taken of the orthonormality of the surface spherical harmonics from (5.1) on the surface of the unit sphere S, this gives

$$W_\delta = \frac{1}{6}\delta^{-3} \sum_{n=2,4,\ldots} \gamma_n^2 \sum_{p=1}^{2n+1} \boldsymbol{\epsilon}^{np} \cdot\cdot \mathbf{C} \cdot\cdot \boldsymbol{\epsilon}^{np} \tag{5.2}$$

The series on the right-hand side of (5.2) is absolutely convergent in view of the real analyticity of the components of the s train tensor in $R^3\backslash 0$. The energy concentration coefficient is determined in a similar manner using (1.3) and (5.2). If Parseval's equality is made use of in (1.1) and (1.3) we obtain (S is a sphere of unit radius in R^3)

$$W_0 = \frac{1}{6}\int_S \hat{\boldsymbol{\epsilon}}(\boldsymbol{\xi}') \cdot\cdot \mathbf{C} \cdot\cdot \hat{\boldsymbol{\epsilon}}(\boldsymbol{\xi}') d\boldsymbol{\xi}' \tag{5.3}$$

An analysis of expressions (3.1) and (3.2) for the strain tensor and formulae (1.1)–(1.3) shows that the accumulated energy function, the strain energy and the energy concentration coefficient turn out to be forms of the fourth power of the vector $\mathbf{n}$ in the case of the zero-moment double force, and of the second power of the vectors $\mathbf{n}$ and $\mathbf{n}_\perp$ in the case of a double force with a moment. Representations of the energy concentration coefficient for a zero-moment double force (a) and a double force with a moment (b) are given below

$$\text{(a) } W_0 = \mathbf{n} \otimes \mathbf{n} \cdots \mathbf{W}_0 \cdots \mathbf{n} \otimes \mathbf{n}, \quad \text{(b) } W_0 = \mathbf{n}_\perp \otimes \mathbf{n} \cdots \mathbf{W}_0 \cdots \mathbf{n} \otimes \mathbf{n}_\perp \tag{5.4}$$

In formula (5.4), $\mathbf{W}_0$ is a symmetric fourth-rank tensor which depends solely on the elastic properties of the anisotropic medium and is written by virtue of expressions (3.1), (3.2) and (5.3) as

$$\mathbf{W}_0 = \frac{2}{3}\pi^2 \int_S \boldsymbol{\xi}' \otimes \hat{\mathbf{E}}(\boldsymbol{\xi}') \otimes \boldsymbol{\xi}' d\boldsymbol{\xi}'$$

where, moreover, account has been taken of the fact that $\hat{\mathbf{A}}(\boldsymbol{\xi}) \cdot \hat{\mathbf{E}}(\boldsymbol{\xi}) = \mathbf{I}$. By using the index form for the components of the stress tensor, it is possible to give the following form to the formula for $\mathbf{W}_0$ which is convenient when performing calculations

$$W_0^{ijkl} = \frac{2}{3}\pi^2 \int_0^\pi \int_0^{2\pi} \xi^i \hat{E}^{jk}(\boldsymbol{\xi}) \xi^l \sin\theta d\theta d\varphi$$

$$\xi_1 = \sin\theta\sin\varphi, \quad \xi_2 = \sin\theta\cos\varphi, \quad \xi_3 = \cos\theta$$

6. ORIENTATION OF DEFECTS IN ANISOTROPIC CRYSTALS

We will now consider the problem of determining the orientation of an isolated defect in the form of a zero-moment double force corresponding to the minimum of the energy concentration coefficient. Defects of this type simulate point edge dislocations; dislocation loops can later arise when such point dislocations combine.

Expression (1.3) shows that the magnitude of the energy concentration coefficient is characterized in a certain manner by the strain energy associated with the formation of such a defect in a crystal. By virtue of (5.4), to determine the minimum of W_0, it is sufficient to calculate the tensor $\mathbf{W}_0$, which is invariant under the different orientations of the defect and then, by convoluting (5.4) with the vectors $\mathbf{n}$, determine the direction of $\mathbf{n}$ for which a minimum of W_0 is attained.

In Figs 1 and 2, graphs of the change in the energy concentration coefficients have been drawn for two crystals: a transversely isotropic zinc crystal and a cubic spinel crystal ($MgAl_2O_4$), respectively. The dimensionless values of the concentration coefficient: $W_0/\| \mathbf{C} \|_2$, where $\| \mathbf{C} \|_2$ is the Euclidean norm of the elasticity tensor, are plotted on the ordinate. These graphs indicate the existence of certain directions in the crystals under consideration for which W_0 attains a minimum value. In the case of the transversely

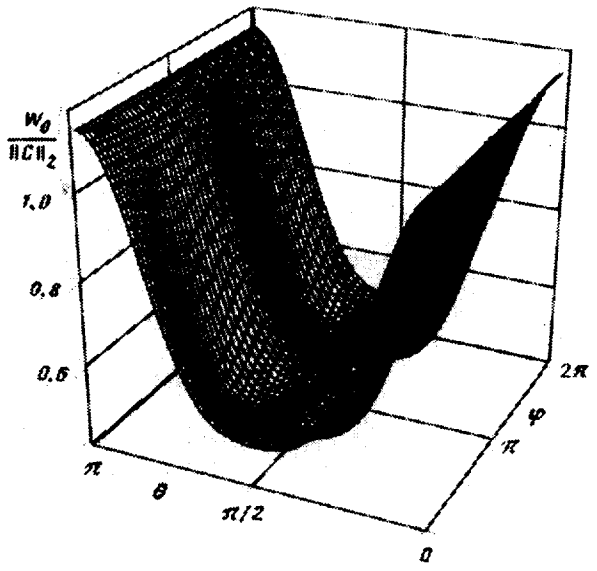

Fig. 1.

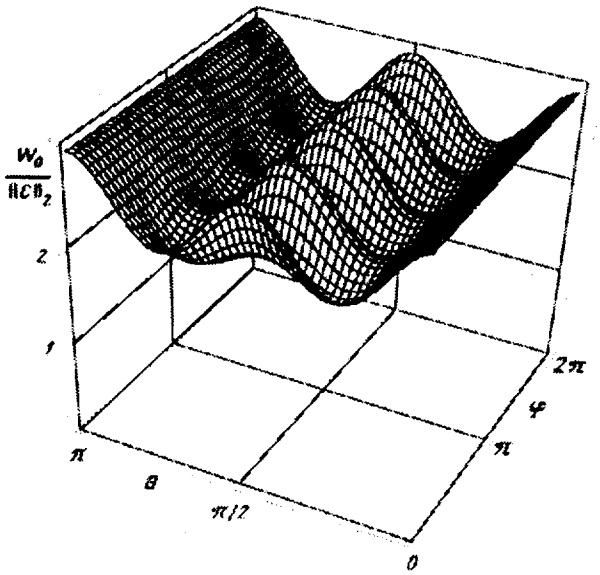

Fig. 2.

isotropic zinc crystal, the directions $\theta = \pi/2$, which correspond to the arrangement of the defect in the form of a zero-moment double force in the plane of isotropy, turn out to be these directions. In the case of the spinel crystal, the minimum directions when $\theta = \pi/4, 3\pi/4$, $\varphi = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ are found to be equally inclined with respect to the vectors of the initial orthogonal basis and coincident with the principal axes of elasticity of the crystal.

To all appearances, the minimum directions correspond to the most probable orientations of defects in the form of a zero-moment double force in the crystals being considered. Analogous results can also be obtained in the case of defects in the form of a double force with a moment.

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